Lecture 1 (45 min + 10 min Q&A)

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Gradient Flows for Shape Optimization in Elasticity

Abstract:
This lecture is devoted to an accessible introduction to shape optimization techniques that combine gradient flow methods with topological derivatives. The primary goal of the lecture is to explain why both tools are needed and how they complement each other in practical optimization problems arising in elasticity.

The lecture starts by recalling the basic idea of shape optimization: one seeks an optimal geometry of a domain that minimizes a given cost functional subject to a governing partial differential equation. Gradient flow methods are introduced as a natural way to iteratively improve the shape by moving the boundary in the direction of steepest descent of the shape functional. These methods are intuitive, stable, and well suited for numerical implementation, but they are limited to smooth deformations that preserve the topology of the domain.

To overcome this limitation, the concept of the topological derivative is introduced. The topological derivative measures the sensitivity of the cost functional with respect to the creation of an infinitesimal hole at a given point in the domain. In the lecture, its role is explained step by step, emphasizing how it provides a criterion for deciding where topological changes are beneficial and how it can be naturally coupled with a gradient flow strategy.

Different choices of shape functionals are then discussed in a pedagogical manner. One option is a Kohn–Vogelius type functional, which compares solutions of elasticity problems corresponding to different boundary conditions and is particularly useful in inverse and identification problems. Another option is based on optimal control problems exhibiting the turnpike property. In this case, students are shown how the long-time behavior of a dynamic problem can be approximated by a steady-state solution, allowing the original evolution equation to be replaced by a static boundary value problem.

The shape functional is defined as the optimal value of the cost associated with this static control problem. The state equation governing the system is the linear elasticity system posed in a bounded domain. Numerical examples are included throughout the lecture to illustrate the theoretical concepts. These examples are based on a Kohn–Vogelius-type formulation arising from inverse problems in elasticity. They demonstrate how gradient flow methods can be used to reconstruct unknown geometries from boundary measurements. In particular, the examples show the progressive evolution of the domain during the optimization process and the monotonic decrease of the mismatch functional along the gradient flow, providing clear insight into the convergence behavior and practical effectiveness of the method.

The lecture concludes with a discussion of the advantages of the proposed approach, highlighting its clear mechanical interpretation, its mathematical consistency, and its suitability for efficient numerical algorithms in structural optimization.